THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280 Introductory Probability 2023-2024 Term 1
Suggested Solutions of Homework Assignment 6

Q1

(a)

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = \int_0^1\int_0^y xdxdy = \frac{1}{6}.$$

(b)

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy = \int_0^1\int_0^y x\frac{1}{y}dxdy = \frac{1}{4}.$$

(c)

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy = \int_0^1\int_0^y y\frac{1}{y}dxdy = \frac{1}{2}.$$

Q2

For $i = 1, 2, \ldots, 1000$, let X_i be the random variable such that $X_i = 1$ if the i-th person gets a card which matches his age, and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{1000} X_i$ is the number of matches. Since for each i, only one of the 1000 cards matches the age of the i-th person, we have

$$E(X_i) = P(X_i = 1) = 1/1000$$

and it follows that

$$E(X) = \sum_{i=1}^{1000} E(X_i) = 1$$

Q3

Define $g(z) = z$ if $z > x$ and $g(z) = 0$ for $z \leq x$. Then $X = g(Z)$ and proposition 2.1 on p. 191 , ch. 5 gives

$$\begin{aligned}
E[X] = E[g(Z)] &= \int_{-\infty}^{\infty} g(z) \cdot f_Z(z) dz \\
&= \int_{-\infty}^{x} 0 \cdot f_Z(z) dz + \int_{x}^{\infty} z \cdot f_Z(z) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} z \cdot e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{x^2}{2}} e^u du \\
&= \frac{1}{\sqrt{2\pi}} \left[e^u \Big|_{-\infty}^{-\frac{x^2}{2}} \right] \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}
\end{aligned}$$

Q4

(a) For $i = 1, 2, \ldots, 365$, let X_i be the random variable such that $X_i = 1$ if the i-th day is a birthday of exactly three people, and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{365} X_i$ is the number of days that are birthdays of exactly three people. Note that for each i,

$$E(X_i) = P(X_i = 1) = \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97}$$

Hence

$$E(X) = \sum_{i=1}^{365} E(X_i) = 365 \cdot \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97} \approx 0.9301$$

(b) For $i = 1, 2, \ldots, 365$, let Y_i be the random variable such that $Y_i = 1$ if the i-th day is the birthday of at least one person, and $Y_i = 0$ otherwise. Then $Y = \sum_{i=1}^{365} Y_i$ is the number of days that are birthdays of at least one person. Note that for each i,

$$E(Y_i) = P(Y_i = 1) = 1 - \left(\frac{364}{365}\right)^{100}$$

Hence

$$E(Y) = \sum_{i=1}^{365} E(Y_i) = 365 \cdot \left(1 - \left(\frac{364}{365}\right)^{100}\right) \approx 87.5755$$

Q5

Note that since X and Y are independent, we have $E(XY) = E(X)E(Y)$. Hence

$$\begin{aligned} E\left((X-Y)^2\right) &= E\left(X^2 - 2XY + Y^2\right) \\ &= E\left(X^2\right) - 2E(XY) + E\left(Y^2\right) \\ &= \mathrm{Var}(X) + E(X)^2 - 2E(X)E(Y) + \mathrm{Var}(Y) + E(Y)^2 \\ &= \sigma^2 + \mu^2 - 2\mu^2 + \sigma^2 + \mu^2 \\ &= 2\sigma^2 \end{aligned}$$

Q6

Note that $\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)$

$$E(XY) = \int_0^\infty \int_0^x xy \cdot \frac{2e^{-2x}}{x} dydx = \frac{1}{4}.$$

$$E(X) = \int_0^\infty \int_0^x x \cdot \frac{2e^{-2x}}{x} dydx = \frac{1}{2}.$$

$$E(Y) = \int_0^\infty \int_0^x y \cdot \frac{2e^{-2x}}{x} dydx = \frac{1}{4}.$$

Hence $\mathrm{Cov}(X, Y) = 1/8$.

Q7

(a) We have

$$E(X) = \sum_{k=1}^{\infty} kP(X = k) = \sum_{k=1}^{\infty} k \left(\frac{5}{6}\right)^{k-1} \frac{1}{6}$$

Using the fact that for $|x| < 1$,

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

it follows that $E(X) = 6$.

(b)

$$\begin{aligned}
E(X \mid Y = 1) &= \sum_{k=1}^{\infty} k \cdot P(X = k \mid Y = 1) \\
&= \sum_{k=2}^{\infty} k \left(\frac{5}{6}\right)^{k-2} \frac{1}{6} \\
&= \sum_{k=1}^{\infty} (1 + k) \left(\frac{5}{6}\right)^{k-1} \frac{1}{6} \\
&= 1 + E(X) \\
&= 7
\end{aligned}$$

(c)

$$\begin{aligned}
E(X \mid Y = 5) &= \sum_{k=1}^{\infty} k \cdot P(X = k \mid Y = 5) \\
&= \sum_{k=1}^{4} k \cdot P(X = k \mid Y = 5) + \sum_{k=6}^{\infty} k \cdot P(X = k \mid Y = 5)
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{4} k \cdot P(X = k \mid Y = 5) &= 1(1/5) + 2(4/5)(1/5) + 3(4/5)^2(1/5) + 4(4/5)^3(1/5) \\
&= \frac{821}{625} \\
\sum_{k=6}^{\infty} k \cdot P(X = k \mid Y = 5) &= \sum_{k=6}^{\infty} k(4/5)^4(5/6)^{k-6}(1/6) \\
&= (4/5)^4 \sum_{k=1}^{\infty} (5 + k)(5/6)^{k-1}(1/6) \\
&= (4/5)^4(5 + E(X)) \\
&= \frac{2816}{625}
\end{aligned}$$

Hence

$$E(X \mid Y = 5) = \frac{3637}{625}$$

Q8

The density of Y is

$$f_Y(y) = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y}dx = e^{-y}, \quad y > 0$$

and $f_Y(y) = 0$ if $y \leq 0$. Hence for $y > 0$,

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{e^{-y}}$$

and

$$E\left(X^2 \mid Y = y\right) = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x \mid y)dx = \int_0^\infty x^2 \frac{e^{-x/y}}{y}dx = 2y^2.$$

Q9

X is a Poisson random variable with parameter $2, Y$ is a binomial random variable with parameter $(10, 3/4)$.
(a)

$$\begin{aligned} P(X+Y=2) &= P(X=0, Y=2) + P(X=1, Y=1) + P(X=2, Y=0) \\ &= P(X=0)P(Y=2) + P(X=1)P(Y=1) + P(X=2)P(Y=0) \\ &= e^{-2}45\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^8 + 2e^{-2}10\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^9 + 2e^{-2}\left(\frac{1}{4}\right)^{10} \\ &\approx 6.027 \times 10^{-5} \end{aligned}$$

(b)

$$\begin{aligned} P(XY=0) &= P(X=0) + P(Y=0) - P(X=0, Y=0) \\ &= e^{-2} + \left(\frac{1}{4}\right)^{10} - e^{-2}\left(\frac{1}{4}\right)^{10} \\ &\approx 0.1353 \end{aligned}$$

(c) $E[XY] = E[X] \cdot E[Y] = 2 \cdot \frac{30}{4} = 15.$

Q10

(a) Note that $E(X_n) = 1$ for each n. Since $\frac{X_n}{3^n} \geq 0$ for each n, by the monotone convergence theorem, we have

$$E(X) = \sum_{n=1}^{\infty} E\left(\frac{X_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{E(X_n)}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$$

(b) Note that $\operatorname{Var}(X_n) = E\left(X_n^2\right) - E(X_n)^2 = 2 - 1^2 = 1$ for each n. Let N be a positive integer. Since $X_1, X_2, \ldots, X_N$ are independent, we have

$$\operatorname{Var}\left(\sum_{n=1}^{N} \frac{X_n}{3^n}\right) = \sum_{n=1}^{N} \frac{\operatorname{Var}(X_n)}{3^{2n}} = \sum_{n=1}^{N} \frac{1}{9^n}$$

Note that

$$\operatorname{Var}\left(\sum_{n=1}^{N} \frac{X_n}{3^n}\right) = E\left(\left(\sum_{n=1}^{N} \frac{X_n}{3^n} - \sum_{n=1}^{N} \frac{1}{3^n}\right)^2\right)$$

converges to $E\left(\left(\sum_{n=1}^{\infty} \frac{X_n}{3^n} - \sum_{n=1}^{\infty} \frac{1}{3^n}\right)^2\right) = \operatorname{Var}\left(\sum_{n=1}^{\infty} \frac{X_n}{3^n}\right)$ as $N \to \infty$ by the dominated convergence theorem. Hence, taking limit $N \to \infty$ in (1), we have

$$\operatorname{Var}\left(\sum_{n=1}^{\infty} \frac{X_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{1}{9^n} = \frac{1}{8}$$

Q11

$$\begin{aligned}
\operatorname{Cov}(X+Y, X-Y) &= E((X+Y)(X-Y)) - E(X+Y)E(X-Y) \\
&= E\left(X^2 - Y^2\right) - (E(X)+E(Y))(E(X)-E(Y)) \\
&= E\left(X^2\right) - E\left(Y^2\right) - \left(E(X)^2 - E(Y)^2\right)
\end{aligned}$$

Since X and Y are identically distributed, it follows that $E(X) = E(Y)$ and $E\left(X^2\right) = E\left(Y^2\right)$. Hence $\operatorname{Cov}(X+Y, X-Y) = 0$.